

CHARACTERIZATION OF INTRINSICALLY HARMONIC FORMS

E. VOLKOV

ABSTRACT. Let M be a closed oriented manifold of dimension n and ω a closed 1-form on it. We discuss the question whether there exists a Riemannian metric for which ω is co-closed. For closed 1-forms with nondegenerate zeros the question was answered completely by Calabi in 1969, cf. [4]. The goal of this paper is to give an answer in the general case, i.e. not making any assumptions on the zero set of ω .

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we will be concerned with the characterization of intrinsically harmonic forms in terms of the topological and smooth structure of the underlying manifold. A k -form on a smooth n -manifold is called intrinsically harmonic if it is closed and its Hodge-dual with respect to some Riemannian metric is also closed. So, for a k -form to be intrinsically harmonic it is necessary to be closed and then the desired Riemannian metric may or may not exist. The question is: given a closed k -form on a smooth manifold, when is it intrinsically harmonic? Let us give a short historical overview. Only the forms of degrees 1 and $n - 1$ have been considered seriously. The forms of degrees strictly between 1 and $n - 1$ seem to present considerable additional difficulties. The following classical theorem of Calabi from 1969 answers the question for forms of degree 1 with nondegenerate zeros.

Theorem 1 (Calabi [4]). *Let ω be a closed 1-form on a closed oriented manifold M . Assume that all the zeros of ω are nondegenerate. Then ω is intrinsically harmonic if and only if it is transitive.*

Transitivity will be discussed in detail later in the paper. For now suffice it to say that a 1-form is called transitive if there exists a closed transversal to its kernel foliation through every point which is not a zero of the form. Note, that one can define the concept of transitivity for $(n - 1)$ -forms repeating verbatim the definition above. In 1996 Honda gave a complete answer to the question for forms with nondegenerate

Date: June 2007.

zeros in degree $n - 1$. Having understood the concept of transitivity for both 1- and $(n - 1)$ -forms we can give a unified formulation of the theorems of Calabi and Honda. We abbreviate forms with nondegenerate zeros as “nondegenerate forms”.

Theorem 2 (Calabi [4], Honda [7]). *Let $k \in \{1, n - 1\}$. For a nondegenerate closed k -form α on a closed oriented connected n -manifold M to be intrinsically harmonic it is necessary and sufficient that*

- (a) *the form α is locally intrinsically harmonic and*
- (b) *the form α is transitive.*

We say that a k -form is locally intrinsically harmonic, if it becomes intrinsically harmonic when restricted to a suitable open neighbourhood of its zero set. For 1-forms with nondegenerate zeros transitivity implies local intrinsic harmonicity and Theorem 2 simplifies to Theorem 1. For the discussion of local intrinsic harmonicity in the case of $(n - 1)$ -forms see the thesis of Honda [7]. All the theorems we have discussed so far assumed that the zeros of the k -form in question are nondegenerate. In 2006 Latschev managed to weaken the assumptions on the zero set of the form.

Theorem 3 (Latschev [8]). *Let $k \in \{1, n - 1\}$. Let α be a closed k -form on a closed oriented connected n -manifold. Assume that the zero set of α is a Euclidean neighbourhood retract. Then α is intrinsically harmonic if and only if the following two conditions are satisfied:*

- (a) *the form α is locally intrinsically harmonic (if $k = 1$ assume in addition that the local metric which provides local harmonicity for α is real analytic) and*
- (b) *the form α is transitive.*

In this paper we stick to the case of 1-forms and prove the characterization theorem in complete generality, i.e without assuming anything on the local structure of the zero set of the 1-form. We also do not assume any regularity higher than C^∞ for the local metric.

Theorem 4. *For a closed 1-form ω on a closed oriented manifold M to be intrinsically harmonic it is necessary and sufficient that*

- (a) *the form ω is locally intrinsically harmonic and*
- (b) *the form ω is transitive.*

For general background we refer to [6].

This work was supported by the DFG grant.

2. PRELIMINARIES

We start by discussing the concept of transitivity in a little more detail.

Definition 1. *A closed 1-form ω is called transitive if for any point $p \in M \setminus S$ there is a closed strictly ω -positive smooth (embedded) path $\gamma : S^1 \longrightarrow M$ through p . Here “strictly ω -positive” means that $\omega(\dot{\gamma}(t)) > 0$ for all $t \in S^1 = \mathbb{R}/\mathbb{Z}$. That is to say that there exists a closed transversal to the kernel foliation of ω through every point of our manifold which does not lie in the zero set of ω .*

Note that if there exists a smooth, not necessarily embedded, strictly ω -positive path through p , then the path is immersed and if $n = \dim M > 2$ we can achieve embeddedness by a small perturbation. If $n = 2$, then we perform the obvious modifications at double points. In the proof below, however, we get embeddedness automatically. To fix conventions from now on “ ω -positive path” means “embedded strictly ω -positive path”.

We recall a classical result from dynamical systems — the Poincaré recurrence theorem.

Proposition 1. *Let (Ω, Σ, μ) be a probability space. Let $\{\phi^t\}_{t \in \mathbb{R}}$ be a measure preserving dynamical system on it. Assume that A is a σ -algebra element of positive measure. Then for any positive N there exists n_0 greater than N such that*

$$\mu(A \cap \phi^{n_0}(A)) > 0.$$

To deal with local questions we will need the following

Definition 2. *For a smooth oriented n -dimensional Riemannian manifold (X, g) we define Laplace-Beltrami operator:*

$$\Delta_g : C^\infty(X) \longrightarrow \Omega^n(X),$$

which converts a smooth function f into a top degree form

$$\Delta_g f := d \star_g df.$$

3. PROOF OF THEOREM 4

For necessity assume there exists a Riemannian metric g which makes ω harmonic. Condition (a) is obviously satisfied. To show Condition (b) we apply the Poincaré recurrence theorem. We set Ω to be our

manifold M , the σ -algebra Σ to be the usual borelian σ -algebra, and μ to be the probability measure defined by a distinguished volume form $dvol$ on M with total volume equal to one. Furthermore, let the vector field X be defined by the following equation: $i_X dvol = \star_g \omega$. Note that X is transverse to the kernel foliation of ω outside S . By the Cartan formula, we see that $L_X dvol = 0$. Let $\{\phi^t\}_{t \in \mathbb{R}}$ be the flow of X on M . In our setting $\{\phi^t\}_{t \in \mathbb{R}}$ becomes a measure preserving dynamical system on (Ω, Σ, μ) . Let now p be a given point in $M \setminus S$. Let $(\bar{\xi}, \Phi)$ be a bi-foliated closed chart around p , i.e. $\bar{\xi}$ is a closed subset of M , containing an open neighbourhood of p and

$$\Phi : \bar{\xi} \longrightarrow B \times I,$$

is a diffeomorphism, where B is a closed ball in \mathbb{R}^{n-1} and $I = [0, 1]$ is a unit time interval. Moreover, under the diffeomorphism Φ flowlines of $\{\phi^t\}_{t \in \mathbb{R}}$ correspond to the vertical leaves $b \times I$, $b \in B$ and integral submanifolds of the kernel foliation of ω correspond to the horizontal leaves $B \times t$, $t \in I$. In further considerations we identify $\bar{\xi}$ with its image under Φ . Since $\bar{\xi}$ is compact, all points of $\bar{\xi}$ will leave it by some time N , as we follow the flow $\{\phi^t\}_{t \in \mathbb{R}}$. We set $A := \bar{\xi}$ and apply Proposition 1 with the above choices of $\Omega, \Sigma, \mu, A, N$. This gives us a trajectory of $\{\phi^t\}_{t \in \mathbb{R}}$ which leaves $\bar{\xi}$ at some point $(b_1, 1)$ and then enters it again for the first time at some point $(b_0, 0)$. Let us denote the flowline between $(b_1, 1)$ and $(b_0, 0)$ by \tilde{c} . It is clear that except for its end points the path \tilde{c} lies outside $\bar{\xi}$. Now we close up this flowline artificially inside the bifoliated chart $\bar{\xi}$, by connecting $(b_0, 0)$ and $(b_1, 1)$ with a smooth path \hat{c} through p , transverse to the horizontal leaves $B \times t$, $t \in I$. Clearly, this can be done in such a way that the concatenation c of the paths \tilde{c} and \hat{c} is smooth and embedded. So as c is a smooth closed ω -positive path and the point p was arbitrary, we have that the form ω is transitive. This is Condition (b).

For sufficiency assume that conditions (a) and (b) hold true. Let U be a neighbourhood of S such that $\omega|_U$ is co-closed with respect to some Riemannian metric g_U on U . It follows from the lemma below that U can be chosen so small that the form $\star_{g_U} \omega_U$ is exact.

Lemma 1. *Let (X, g) be a smooth oriented n -dimensional Riemannian manifold without boundary. Let S be a compact zero set of a 1-form γ on X which is both closed and co-closed. There exists an open neighbourhood U of S , such that for any closed $(n-1)$ -form ψ on X the restriction $\psi|_U$ is exact.*

Proof. The form γ is a solution to a first order linear elliptic equation

$$(1) \quad (d + d^*)\gamma = 0,$$

where $d + d^* = d + \star d \star$ is a Dirac operator on X . Locally (1) is equivalent to $\Delta_g f = 0$, where f is a local primitive function of γ . So, we can apply the result by Aronszajn, cf. [1], to get that the Dirac operator on 1-forms possesses the strong unique continuation property. Then we apply the theorem by C. Bär (cf. [2]) to find a sequence $\{L_k\}_{k \in \mathbb{N}}$ of submanifolds of X of codimension at least 2, with $S \subset \bigcup_{k \in \mathbb{N}} L_k$. Since every submanifold S_k can be countably exhausted by compact ones (possibly with boundary), we may without loss of generality assume that each L_k is compact, possibly with boundary. Set $Z_k = S \cap L_k$. Let \dim denote the covering dimension of a topological space. Then $\dim Z_k \leq n - 2$, since L_k is a compact manifold (possibly with boundary) of dimension at most $n - 2$ and $Z_k \subset L_k$. Since $S = \bigcup_{k \in \mathbb{N}} Z_k$ and every Z_k is closed in S , the Countable Sum Theorem (cf. [5] Theorem 7.2.1 on the page 394) implies that $\dim S \leq n - 2$. This, in turn, implies that $H_{\check{C}ech}^{n-1}(S) = 0$.

Take a sequence $\{U_j\}_{j \in \mathbb{N}}$ of open neighbourhoods U_j of S such that $U_{j+1} \subset U_j$ and $\bigcap_{j \in \mathbb{N}} U_j = S$ with $U_0 = X$. The continuity property of Čech cohomology, cf. [3] (section 14 “Continuity”, Theorem 14.4), implies that $\varinjlim_{\check{C}ech} H_{\check{C}ech}^{n-1}(U_j) = 0$, but U_j is a manifold, hence Čech cohomology of it is the same as de Rham and finite dimensional. So we have that a direct limit of a sequence of finite dimensional vector spaces $H_{\check{C}ech}^{n-1}(U_j)$ vanishes. This implies that for j large enough the image of the 0-th vector space of the sequence in the j -th one vanishes. In other words if $i : U_j \rightarrow X$ denotes the obvious inclusion, then $i^* H^{n-1}(X)$ is the trivial subspace of $H^{n-1}(U_j)$. Take $U := U_j$. \square

So, we can pick a primitive $(n-2)$ -form α on U : ($d\alpha = \star_{g_U} \omega|_U$). Using transitivity of the form ω , by a standard “thickening of a transversal argument” (see for example [6]) we obtain that given a point $m \in M \setminus S$, there exists an open neighbourhood W_m of it, diffeomorphic to $S^1 \times B$, where B is an open ball in \mathbb{R}^{n-1} centered at the origin. Moreover, when restricted to W_m , the form ω is proportional to $d\theta$, where θ denotes the coordinate along the S^1 direction. Let $\rho : [0, 1] \rightarrow \mathbb{R}$ be a smooth cut-off function: $\rho|_{[0, 1/5]} = 1$, $\rho|_{[4/5, 1]} = 0$. Set $\psi_m = \rho(x_1^2 + \dots + x_{n-1}^2) dx_1 \wedge \dots \wedge dx_{n-1}$. Clearly, the $(n-1)$ -form ψ_m is closed, vanishes in a neighbourhood of the boundary of P and the top degree form $\Theta := \omega \wedge \psi_m$ satisfies the following properties: Θ is non-negative everywhere and $\Theta > 0$ in some neighbourhood V_m of $\gamma(S^1)$. Vanishing of ψ_m near the boundary of P implies that ψ_m vanishes in some open neighbourhood U_m of S with $U_m \subset U$. This construction almost literally follows the one given by Calabi in [4].

Since $M \setminus U$ is compact it can be covered by V_{m_1}, \dots, V_{m_l} for some natural number l , where $m_1, \dots, m_l \in M \setminus U$. Set

$$U_0 := U_{m_1} \cap \dots \cap U_{m_l},$$

$$V := V_{m_1} \cup \dots \cup V_{m_l}$$

and

$$\psi' := \sum_{i=1}^l \psi_{m_i}.$$

Note, that $U_0 \subset M \setminus V \subset U$ and $\psi'|_{U_0} = 0$.

We pause for a moment to summarize what we have. We have an open neighbourhood U of S with an $(n-2)$ -form α on U such that $d\alpha = \star_{g_U} \omega$; open sets U_0 and V with $U_0 \subset M \setminus V \subset U$ and an $(n-1)$ -form ψ' with $\psi' \wedge \omega$ bounded away from zero on V , nonnegative everywhere and satisfying

$$\psi'|_{U_0} = 0.$$

This allows us to finish the proof with the standard gluing argument. We let ϕ be a smooth function with $\phi|_{M \setminus V} = 1$ and $\phi|_{M \setminus U} = 0$. Such a function ϕ exists since both sets $M \setminus V$ and $M \setminus U$ are closed and the first one is contained in the complement of the second. Set $\alpha'' = \phi\alpha$ and $\psi'' = d\alpha''$. Note that $\psi''|_{M \setminus V} = d\alpha|_{M \setminus V} = \star_{g_U} \omega|_{M \setminus V}$. Consider a closed form

$$\psi = K\psi' + \psi''$$

for sufficiently large positive constant K . We claim that the form ψ has the following properties:

- (i) $\psi|_{U_0} = \star_{g_U} \omega|_{U_0}$,
- (ii) $\omega \wedge \psi > 0$ everywhere on $M \setminus S$.

Indeed, since $\psi'|_{U_0} = 0$, we have that

$$\psi|_{U_0} = \psi''|_{U_0} = \star_{g_U} \omega|_{U_0}.$$

This shows the first property. For the second one consider

$$\psi|_{M \setminus V} = K\psi'|_{M \setminus V} + \psi''|_{M \setminus V} = K\psi'|_{M \setminus V} + \star_{g_U} \omega|_{M \setminus V},$$

multiplying with ω gives us

$$\omega \wedge \psi|_{M \setminus V} = K\omega \wedge \psi'|_{M \setminus V} + \omega \wedge \star_{g_U} \omega|_{M \setminus V}.$$

The last expression is the sum of two nonnegative terms, the second one being strictly positive outside S . We are left the expression $\omega \wedge \psi$, restricted to V . Since $\omega \wedge \psi'|_V$ is bounded away from zero, we have that

$$\omega \wedge \psi|_V = K\omega \wedge \psi'|_V + \omega \wedge \psi''|_V > 0$$

for sufficiently large positive constant K .

Now, having the form ψ with the properties above we construct the desired metric g by gluing. Let ϕ_U, ϕ_V be a partition of unity, subordinate to the cover U, V . Let g'' be any metric on V , making ω and ψ orthogonal to each other. Consider the metric $\tilde{g} := \phi_U g_U + \phi_V g''$ on M . It makes ω and ψ orthogonal everywhere on M and $\star_{\tilde{g}} \omega|_{U_0} = \star_{g_U} \omega|_{U_0} = \psi|_{U_0}$. Consider the following orthogonal decomposition of the tangent bundle of $M \setminus S$:

$$\tilde{g} = \tilde{g}|_{\text{Ker}\omega} \oplus \tilde{g}|_{\text{Ker}\psi}.$$

There exists a unique smooth function $\tilde{f} : M \setminus S \rightarrow \mathbb{R}$, such that for the metric

$$g := \tilde{f} \tilde{g}|_{\text{Ker}\omega} \oplus \tilde{g}|_{\text{Ker}\psi}$$

on $M \setminus S$ we have that $\star_g \omega|_{M \setminus S} = \psi|_{M \setminus S}$. Note, that $\tilde{f}|_{U_0} = 1$, therefore $g|_{U_0 \setminus S} = \tilde{g}|_{U_0 \setminus S}$, and hence g can be C^∞ -regularly continued across points of S by just setting $g|_S := \tilde{g}|_S$. This means that the metric g is well-defined everywhere on M . The equation

$$\star_g \omega = \psi$$

holds on $M \setminus S$, by the choice of \tilde{f} and it also holds on U_0 , by the first property of the form ψ because $g|_{U_0} = \tilde{g}|_{U_0} = g_U|_{U_0}$. Thus, since the form ψ is closed we obtain that the form ω is co-closed with respect to the metric g everywhere on M . This completes the proof of Theorem 4.

4. CONCLUDING REMARKS.

As we mentioned in the introduction the forms of degrees strictly between 1 and $n-1$ present considerable additional difficulties. Indeed, the simplest case of such a form would be a 2-form on a 4-manifold. A generic 2-form on a 4-manifold does not have any zeros at all. So let α be a nowhere zero closed 2-form on a 4-manifold. To simplify things even further assume that α has constant rank. For dimension reasons we have only two possibilities for the rank of α — 2 or 4. In the last case the form α is symplectic, and therefore is harmonic for any metric g which is compatible with α . The question of intrinsic harmonicity is answered trivially and positively in this case. So the only potentially interesting case is when α has constant rank 2. It turns out that this case presents serious difficulties. As the following example shows, transitivity is not sufficient for harmonicity.

Example 1. Let M be the total space of the nontrivial S^2 -bundle $\xi = (S^2 \rightarrow M \xrightarrow{\pi} S^2)$ over S^2 . It is easy to see that there exists a section s of ξ through every point of M . Let $dvol_{S^2}$ be a volume

form on the base S^2 and set $\alpha := \pi^* dvol$. The form α is a closed 2-form of constant rank 2 on the 4-dimensional manifold M , where the fibers of ξ are the leaves of the 2-dimensional kernel foliation of α . Sections of ξ provide closed 2-dimensional submanifolds of M to which α restricts as a volume form, so α is transitive. But α is not (!) intrinsically harmonic. Assume by contradiction, that there exists a Riemannian metric g on M such that the form $\psi := \star_g \alpha$ is closed. The form ψ has constant rank 2 and the leaves of the kernel foliation of ψ are transverse to those of α , i.e. to the fibers of ξ . Take any leaf \mathcal{L} of the kernel foliation of ψ . The restriction $\pi_{\mathcal{L}} : \mathcal{L} \rightarrow S^2$ is a submersion and therefore for dimension reasons a covering map. So \mathcal{L} is diffeomorphic to S^2 . So the total space M of ξ admits a foliation by closed leaves transverse to the fibers with every leaf intersecting every fiber exactly once contradicting the nontriviality of ξ .

Tautologically one can say that a closed 2-form of rank 2 on a 4-manifold is intrinsically harmonic if and only if its kernel foliation $\text{Ker} \omega$ admits a complementary foliation \mathcal{F} defined by a closed 2-form ψ . Indeed, given g which makes ω harmonic, we can set $\psi := \star_g \omega$. Conversely, given a closed 2-form ψ defining a foliation \mathcal{F} complementary to $\text{Ker} \omega$ we can define a Riemannian metric g by requiring that $\text{Ker} \omega$ and \mathcal{F} are orthogonal. Then $\star_g \omega$ is proportional to ψ . By rescaling g on $T\text{Ker} \omega$ we can achieve that $\star_g \omega$ is equal to ψ , which is closed. The big problem with this characterization is that deciding the existence of a complementary foliation defined by a closed form is just as hard as deciding whether the given form is intrinsically harmonic. Finally note that all this is not even beginning to touch the case of nonconstant rank.

REFERENCES

- [1] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, J. Math. Pures Appl. (9) 36 (1957) pp. 235-249.
- [2] C. Bär, Zero Sets of Solutions to Semilinear Elliptic Systems of First Order, Invent. Math., 138 (1999), no. 1, pp. 183-202.
- [3] G. Bredon, Sheaf Theory, McGraw-Hill Inc. 1967.
- [4] E. Calabi, An intrinsic characterization of harmonic one-forms, Global Analysis, Papers in Honour of K. Kodaira, 1969, D.C. Spencer and S. Iyanaga editors, pp. 101-107 Univ. Tokyo Press, Tokyo.
- [5] R. Engelking, General Topology, Helderman Verlag, 1989.
- [6] M. Farber, Topology of closed One-forms, Mathematical Surveys and Monographs vol. 108, American Mathematical Society 2004.
- [7] K. Honda, On Harmonic forms for Generic Metrics, PhD Thesis, Princeton University, Princeton, 1996.

- [8] J. Latschev, Closed forms transverse to singular foliations, Manuscripta Math. 121, 293-315 (2006).

MATH. INST. LMU, THERESIENSTR. 39, 80333 MÜNCHEN, GERMANY
E-mail address: volkov@mathematik.uni-muenchen.de